EEAP 246
BRANICKY

SINUSOIDAL STEADY-STATE RESPONSE

→ SINUSOIDAL INPUTS USED TO CHARACTERIZE THE RESPONSE OF A SYSTEM

- SPEAKERS, AMPLIFIERS, + OTHER AUDIO COMPONENTS
- AUDIO TAPE, VIDEO TAPE, + OTHER RECORDING MEDIA
- ELECTRICAL AND MECHANICAL SYSTEMS

→ WHAT IS THE RESPONSE OF AN LTI SYSTEM TO A COMPLEX SINUSOID INPUT?

IMPULSE RESPONSE IS $h(t)$
INPUT IS $x(t) = e^{j\omega t}$

CONVOLUTION INTEGRAL SAYS

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau \\
&= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \\
&= H(j\omega)\, e^{j\omega t}
\end{aligned}
$$

WHERE $\quad H(j\omega) \triangleq \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$ — FREQ. RESPONSE OF SYSTEM

$$y(t) = \underbrace{|H(j\omega)|}_{\text{MODIFICATION OF AMPLITUDE}}\, e^{j(\omega t + \underbrace{\angle H(j\omega)}_{\text{MODIFICATION OF PHASE}})}$$

Computer Shopper.com

Sonic
Superstars

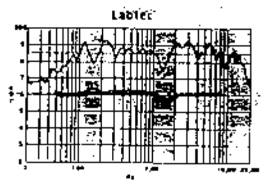
(Continued)

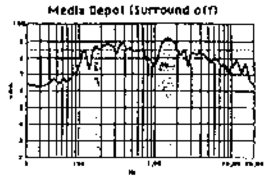
Frequency Response

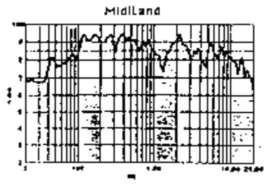
by Don Labriola
Originally published in the September 1997 issue of Computer Shopper

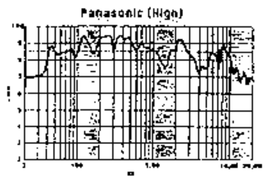
The frequency response curve for each speaker illustrates its ability to reproduce sounds at their correct volumes across the audible frequency range, represented on the horizontal axis. Because we calibrated each speaker system to play at 85dBr, a perfect frequency-response graph would be a straight horizontal line at 85dBr. Higher dBr values indicate louder performance.

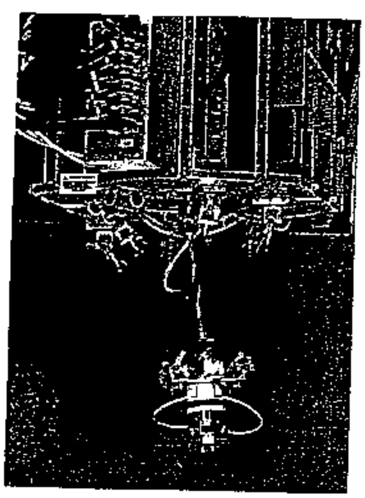
Read below for specs on Labtec, Media Depot, MidiLand and Panasonic speakers. The previous page shows details on 4Q, Altec Lansing, Benwin, Jazz and Just products.

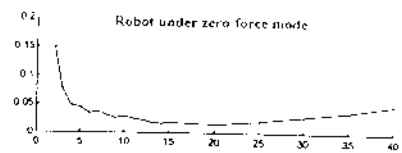


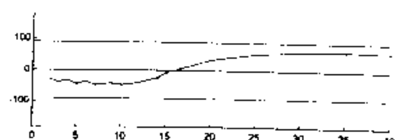




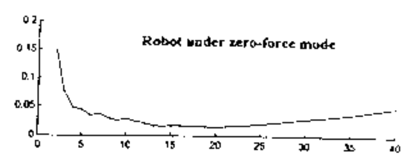


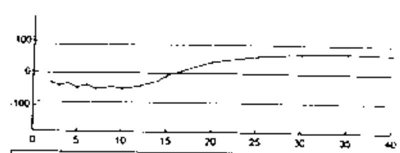






Adminance response obtained by doing a least square sinusoidal fit to the actual data





Admittance response obtained by taking the DFT of the actual data and extracting the response to the fundamental frequency component

WHAT ABOUT REAL-VALUED SINUSOIDS?

USE INPUT
$$x(t) = A\cos(\omega t + \varphi) = \frac{A}{2} e^{j(\omega t+\varphi)} + \frac{A}{2} e^{-j(\omega t+\varphi)}$$

LINEARITY MEANS
$$y(t) = |H(j\omega)| \frac{A}{2} e^{j(\omega t + \varphi + \angle H(j\omega))} + |H(-j\omega)| \frac{A}{2} e^{-j(\omega t + \varphi - \angle H(-j\omega))}$$

NOTE: $h(t)$ REAL $\Rightarrow$ $|H(j\omega)|$ IS EVEN
$\angle H(j\omega)$ IS ODD

$$\Longrightarrow \quad y(t) = \underbrace{|H(j\omega)|}_{\text{MODIFICATION OF MAG.}} A \cos\big(\omega t + \varphi + \underbrace{\angle H(j\omega)}_{\text{MODIFICATION OF PHASE}}\big)$$

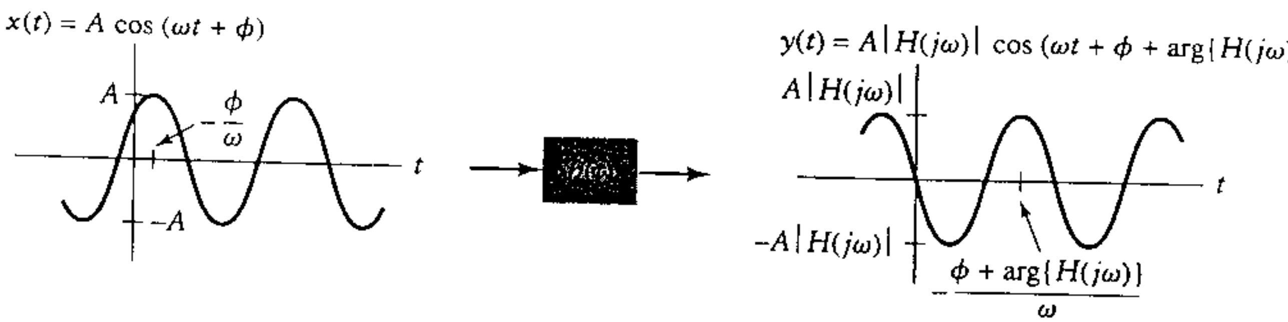
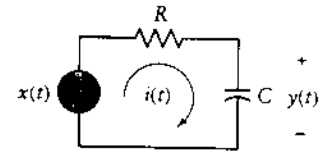


FIGURE 2.18 A sinusoidal input to a LTI system results in a sinusoidal output of the same frequency with the amplitude and phase modified by the system's frequency response.

$\rightarrow$ THIS IS EASILY MEASURED WITH A SINUSOIDAL OSCILLATOR + AN OSCILLOSCOPE

Example 2.15 The impulse response of the system relating the input voltage to the voltage across the capacitor in Fig. 2.10 is given by

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$



Find an expression for the frequency response and plot the magnitude and phase response.

Solution: Substituting $h(t)$ into Eq. (2.27) gives

$$\begin{aligned} H(j\omega) &= \frac{1}{RC}\int_{-\infty}^{\infty} e^{-\tau/RC}u(\tau)e^{-j\omega\tau}\,d\tau \\ &= \frac{1}{RC}\int_{0}^{\infty} e^{-(j\omega+1/RC)\tau}\,d\tau \\ &= \frac{1}{RC}\frac{-1}{(j\omega+1/RC)}e^{-(j\omega+1/RC)\tau}\Big|_0^{\infty} \\ &= \frac{1}{RC}\frac{-1}{(j\omega+1/RC)}(0-1) \\ &= \frac{1/RC}{j\omega+1/RC} \end{aligned}$$

The magnitude response is

$$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$

while the phase response is

$$\arg\{H(j\omega)\} = -\arctan(\omega RC)$$

The magnitude response and phase response are presented in Figs. 2.20(a) and (b), respectively. The magnitude response indicates that the RC circuit tends to attenuate high-frequency sinusoids. This agrees with our intuition from circuit analysis. The circuit cannot respond to rapid changes in the input voltage. High-frequency sinusoids also experience a phase shift of $\pi/2$ radians. Low-frequency sinusoids are passed by the circuit with much higher gain and experience relatively little phase shift.

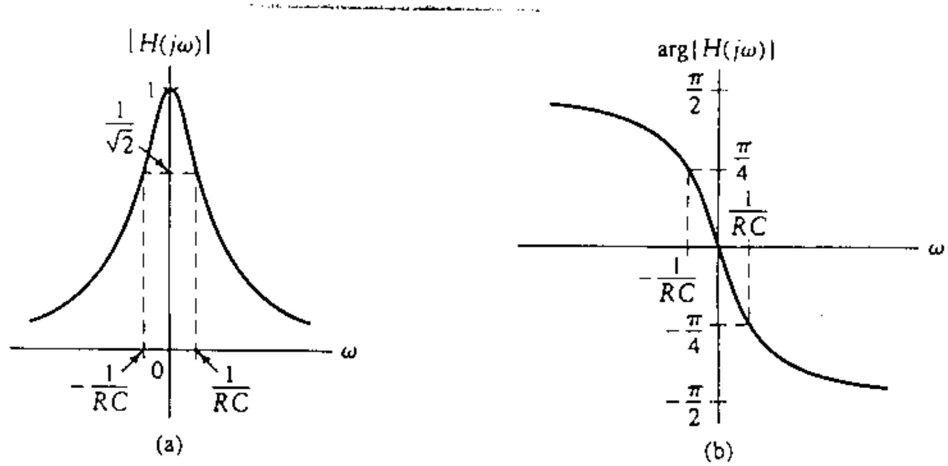


Figure 2.20 Frequency response of the RC circuit in Fig. 2.10. (a) Magnitude response. (b) Phase response.

SOURCE: HAYKIN AND VAN VEEN'S TEXT

FREQUENCY-DOMAIN ANALYSIS OF SYSTEMS

- ☐ BASICS
- ☐ FILTERING
- ☐ MODULATION
- ☐ SAMPLING

BASICS

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{GENERAL LTI}\\ h(t)\end{array}} \longrightarrow y(t), \qquad y(t) = h(t) * x(t)$$

TIME-DOMAIN VIEW

$$\updownarrow$$

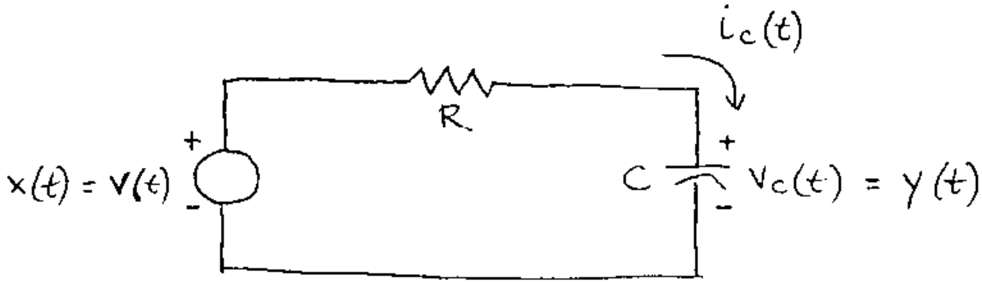
$$X(\omega) \longrightarrow \boxed{\begin{array}{c}\text{GENERAL LTI}\\ H(\omega)\end{array}} \longrightarrow Y(\omega), \qquad Y(\omega) = H(\omega)X(\omega)$$

FREQUENCY-DOMAIN VIEW

$$x(t) \longleftrightarrow X(\omega)$$
$$h(t) \longleftrightarrow Y(\omega)$$
$$y(t) \longleftrightarrow Y(\omega)$$

THUS $y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t}\,d\omega$

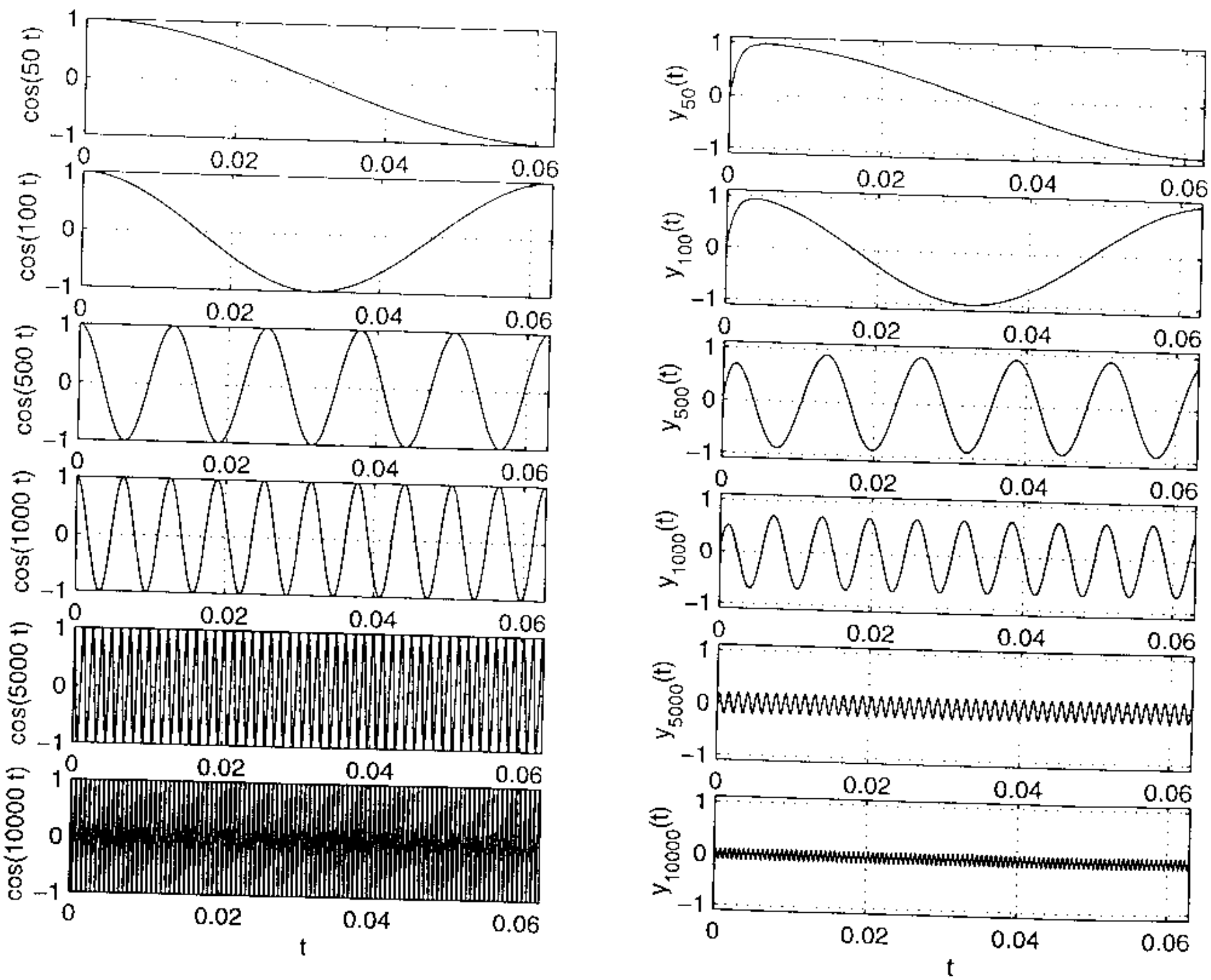
AN EXAMPLE



GIVEN: $1/RC = 1000$

Q: WHAT IS CIRCUIT'S RESPONSE TO $x(t) = \cos(\omega_0 t)$?

A1: TEST ITS RESPONSE FOR A SAMPLING OF INPUTS



YOU MIGHT NOTICE THAT

"The response to a sinusoidal input is also a sinusoid, having the same frequency as the input, but scaled in amplitude and phase shifted."

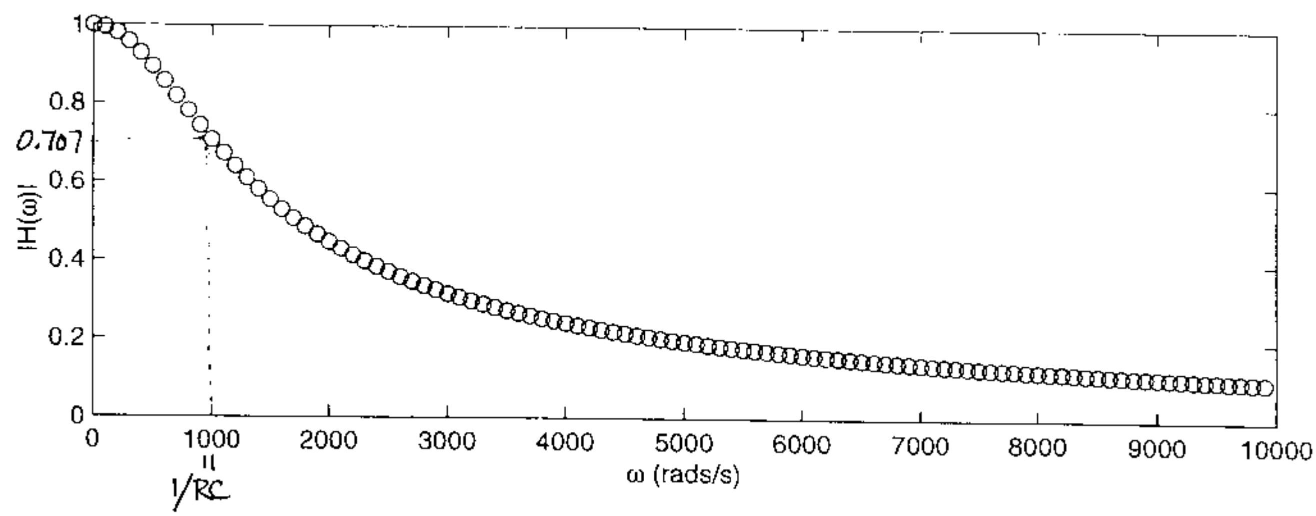
THEREFORE, IT IS ENOUGH TO RECORD

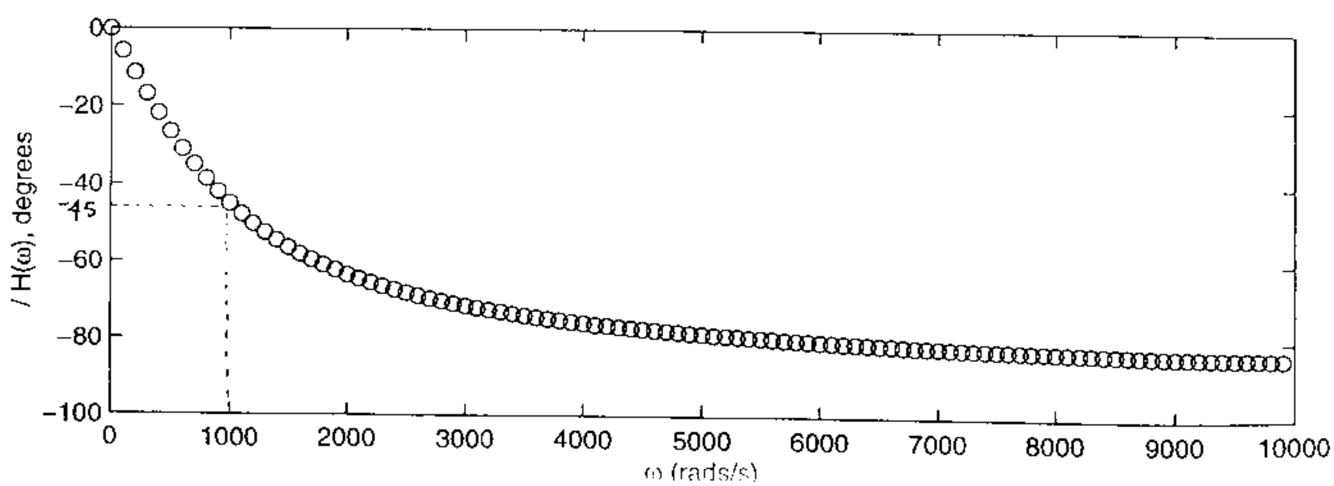
① THE MAGNITUDE SCALING FACTOR

② THE PHASE SHIFT

FOR EACH TEST FREQUENCY.

BACK TO THE LAB ...





COULD YOU HAVE PREDICTED THIS?

A2: KVL:
$$v(t) = v_R(t) + v_c(t)$$
$$x(t) = R i_c(t) + y(t)$$

CAPACITOR:
$$i_c(t) = C \frac{dv_c(t)}{dt} = C \frac{dy(t)}{dt}$$

COMBINING, WE CAN EXPRESS OUTPUT IN TERMS OF INPUT:

$$x(t) = RC \frac{dy(t)}{dt} + y(t)$$

$$\frac{dy(t)}{dt} = -\frac{1}{RC} y(t) + \frac{1}{RC} x(t) \qquad (\star)$$

$(\star)$ IS OF FORM	$\dot{y} = a y + b x$
SOLN IS	$y(t) = e^{at} y(0) + \int_0^t e^{a(t-\lambda)} b x(\lambda)\, d\lambda$

SOLN IS
$$y(t) = e^{-\alpha t} y(0) + \int_0^t e^{-\alpha(t-\lambda)} \alpha\, x(\lambda)\, d\lambda$$
$$\alpha = 1/RC$$

FOR OUR TESTS, $x(t)$ IS APPLIED WITH NEGLIGIBLE VOLTAGE ON THE CAPACITOR, SO...

$$y(t) = \int_0^t e^{-\alpha(t-\lambda)} \alpha\, x(\lambda)\, d\lambda$$

NOW, JUST PLUG IN $x(\lambda) = \cos(\omega_0 \lambda)$ AND INTEGRATE.

TABLE OF INTEGRALS:
$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$$

$$y(t) = \alpha e^{-\alpha t} \left[\frac{e^{\alpha \lambda}}{\alpha^2 + \omega_0^2} \left(\alpha \cos \omega_0 \lambda + \omega_0 \sin \omega_0 \lambda \right) \right]_{\lambda = 0}^{t}$$

$$= \frac{\alpha e^{-\alpha t}}{\alpha^2 + \omega_0^2} \left[e^{\alpha t} \left(\alpha \cos \omega_0 t + \omega_0 \sin \omega_0 t \right) - \alpha \right]$$

$$= \frac{\alpha}{\alpha^2 + \omega_0^2} \cdot \sqrt{\alpha^2 + \omega_0^2} \cos\left(\omega_0 t - \tan^{-1}(\omega_0 / \alpha) \right) - \frac{\alpha^2 e^{-\alpha t}}{\alpha^2 + \omega_0^2}$$

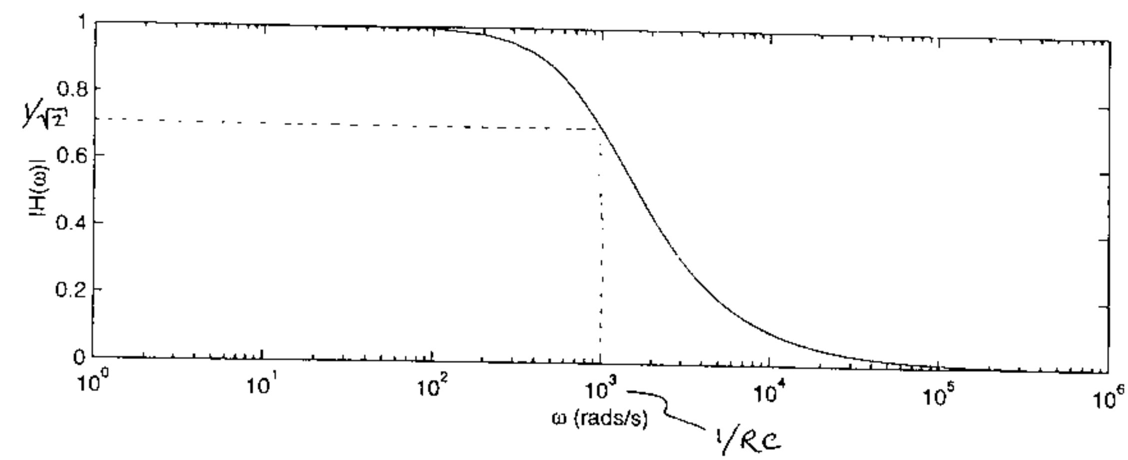
TRIG IDENTITY:

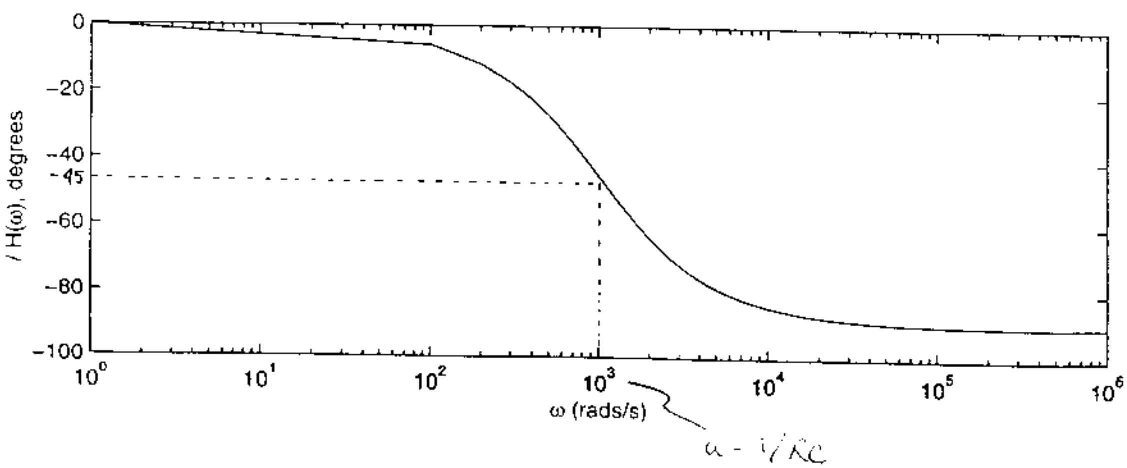
$$\boxed{A \cos(x) + B \sin(x) = \sqrt{A^2 + B^2} \cos\left(x - \tan^{-1}(B/A) \right)}$$

$$y(t) \approx \frac{\alpha}{\sqrt{\alpha^2 + \omega_0^2}} \cos\left(\omega_0 t - \tan^{-1}\left(\omega_0 / \alpha \right) \right)$$

MAGNITUDE SCALING VS. ω : $\dfrac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$

PHASE SHIFT VS. ω : $-\tan^{-1}(\omega / \alpha)$





COULD YOU HAVE DONE THIS ANOTHER WAY?

A3: $y(t) = \int_{-\infty}^{t} e^{-\alpha(t-\lambda)} \alpha \, x(\lambda) \, d\lambda$

CHOOSE COMPLEX EXPONENTIAL AT FREQUENCY ω_0 AS TEST SIGNAL: $x(t) = e^{j\omega_0 t}$

$$
\begin{aligned}
y(t) &= \alpha e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha\lambda} e^{j\omega_0 \lambda} d\lambda \\
&= \alpha e^{-\alpha t} \left[\frac{1}{\alpha + j\omega_0} e^{(\alpha + j\omega_0)\lambda} \right]_{\lambda=-\infty}^{t} \\
&= \alpha e^{-\alpha t} \frac{1}{\alpha + j\omega_0} e^{(\alpha + j\omega_0)t} \\
&= \frac{\alpha}{\alpha + j\omega_0} e^{j\omega_0 t} \\
&= H(\omega_0) e^{j\omega_0 t}
\end{aligned}
$$

$$|H(\omega_0)| = \frac{\alpha}{\sqrt{\alpha^2 + \omega_0^2}}$$

$$\angle H(\omega_0) = 0 - \tan^{-1}(\omega_0/\alpha)$$

$\therefore$ COMPLEX SINUSOID AT SAME FREQ., BUT SCALED/SHIFTED (AMPLITUDE/PHASE)

WHAT IS RESPONSE TO $\cos \omega_0 t$?

$$\cos \omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

WHAT IS RESPONSE TO ANY PERIODIC SIGNAL (WITH $T = \frac{2\pi}{\omega_0}$)?

$$x(t) = \sum_{n=-\infty}^{\infty} c_{n,x} e^{jn\omega_0 t} \Rightarrow y(t) = \sum_{n=-\infty}^{\infty} \underbrace{H(n\omega_0) c_{n,x}}_{c_{n,y}} e^{jn\omega_0 t}$$

WHAT IS RESPONSE TO ANY SIGNAL?

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \Longrightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) X(\omega) e^{j\omega t} d\omega$$