Problem Set #8 Solutions

M.S. Branicky

Problem 8.1

Bayes' rule gives that

$$P(Bi|R) = P(R|Bi)P(Bi)/P(R).$$

Note that this also means that

$$P(Bi|R) = \alpha P(R|Bi)P(Bi) = \alpha P(R, Bi).$$

The conditional probabilities of R given the box can be computed directly from the data:

$$\begin{aligned} P(R|B1) &= 2/(2+3+6) = 2/11, \\ P(R|B2) &= 4/(4+2+3) = 4/9, \\ P(R|B3) &= 3/(3+4+3) = 3/10. \end{aligned}$$

Each box is equally likely, so $P(Bi) = 1/3$ for each $i = 1, 2, 3$. Thus,

$$\begin{aligned} P(R, B1) &= P(R|B1)P(B1) = (2/11)(1/3) = 2/33 = && 180/2970, \\ P(R, B2) &= P(R|B2)P(B2) = (4/9)(1/3) = 4/27 = && 440/2970, \\ P(R, B3) &= P(R|B3)P(B3) = (3/10)(1/3) = 1/10 = && 297/2970. \end{aligned}$$

Now, one can either calculate $P(R)$ by finding the marginal distribution over the above,[1] or normalize because we have

$$\begin{aligned} P(B1|R) &= \alpha 180, \\ P(B2|R) &= \alpha 440, \\ P(B3|R) &= \alpha 297, \end{aligned}$$

and know that these add up to 1 (since they cover all possibilities conditioned on the same event).

$$\boxed{\begin{aligned} P(B1|R) &= 180/(180+440+297) = && 180/917 = 0.1963 \\ P(B2|R) &= 440/(180+440+297) = && 440/917 = 0.4798 \\ P(B3|R) &= 297/(180+440+297) = && 297/917 = 0.3239 \end{aligned}}$$

A common mistake is to compute $P(R) = 9/30 = 3/10$ because 9 of the total of 30 balls are red. This s wrong and will lead to probabilities above that add to slightly more than 1.[2]

[1] Adding over all three cases for Bi yields $P(R) = 917/2970 = 0.3088$

[2] To see why, suppose there were only two boxes and two colors of balls and that the numbers had been a little different: $B1$, 98 red balls and 1 white ball; $B2$, 0 red balls and 1 white ball. Then, the naïve counting method yields $P(R) = 98/100$, when the probability is actually less than one half.

Problem 8.2

Again, using Bayes' rule, we know that

$$P(a|d) = P(d|a)P(a)/P(d).$$

or (for purposes of normalization)

$$P(a|d) = \alpha P(d|a)P(a) = \alpha P(a,d).$$

We first compute the marginal $P(a,d)$:

$$\begin{aligned}
P(a,d) &= \sum_B \sum_C P(a,B,C,d) \\
&= \sum_B \sum_C P(d|B,C)P(B|a)P(C|a)P(a) \\
&= (1/2)\sum_B \sum_C P(d|B,C)P(B|a)P(C|a) \\
&= (1/2)[P(d|b,c)P(b|a)P(c|a) + P(d|b,\neg c)P(b|a)P(\neg c|a) + \\
&\qquad P(d|\neg b,c)P(\neg b|a)P(c|a) + P(d|\neg b,\neg c)P(\neg b|a)P(\neg c|a)] \\
&= (1/2)[(1)(1)(1) + (1/2)(1)(0) + (1/2)(0)(1) + (0)(0)(0)] \\
&= 1/2
\end{aligned}$$

Thus,

$$P(a|d) = \alpha(1/2) \tag{1}$$

Likewise,

$$\begin{aligned}
P(\neg a,d) &= \sum_B \sum_C P(\neg a,B,C,d) \\
&= \sum_B \sum_C P(d|B,C)P(B|\neg a)P(C|\neg a)P(\neg a) \\
&= (1/2)\sum_B \sum_C P(d|B,C)P(B|\neg a)P(C|\neg a) \\
&= (1/2)[P(d|b,c)P(b|\neg a)P(c|\neg a) + P(d|b,\neg c)P(b|\neg a)P(\neg c|\neg a) + \\
&\qquad P(d|\neg b,c)P(\neg b|\neg a)P(c|\neg a) + P(d|\neg b,\neg c)P(\neg b|\neg a)P(\neg c|\neg a)] \\
&= (1/2)[(1)(1/2)(1/2) + (1/2)(1/2)(1/2) + (1/2)(1/2)(1/2) + (0)(1/2)(1/2)] \\
&= 1/4
\end{aligned}$$

Thus,

$$P(\neg a|d) = \alpha(1/4) \tag{2}$$

Combining Equations (1) and (2), we obtain

$$P(a|d) = (1/2)/(1/2 + 1/4) = \boxed{2/3}$$

Problem 8.3

First note, that Equation (14.1) of Russell & Norvig plus the Bayes' net imply that

$$P(J, M, A, B, E) = P(J|A)P(M|A)P(A|B, E)P(B)P(E)$$

Now, we are asked to compute $P(\neg j, \neg m, b, e)$, which is a marginal probability, because it does not involve the variable A for the Alarm. Thus, we must sum over values of A:

$$\begin{aligned}
P(\neg j, \neg m, b, e) &= \sum_A P(\neg j, \neg m, A, b, e) \\
&= \sum_A P(\neg j|A)P(\neg m|A)P(A|b, e)P(b)P(e) \\
&= P(b)P(e) \sum_A P(\neg j|A)P(\neg m|A)P(A|b, e) \\
&= P(b)P(e)[P(\neg j|a)P(\neg m|a)P(a|b, e) + P(\neg j|\neg a)P(\neg m|\neg a)P(\neg a|b, e)] \\
&= (0.001)(0.002)[(0.05)(0.30)(0.98) + (0.99)(0.99)(0.02)] \\
&= (0.001)(0.002)(0.034302) \\
&= \boxed{6.8604 \times 10^{-8}}
\end{aligned}$$

Problem 8.4

The Bayes' network and CPTs are

$$\boxed{\text{BatteryLow (B)}} \longrightarrow \boxed{\text{OrvilleDropsBall (D)}} \longrightarrow \boxed{\text{ObserverSaysDrops (O)}}$$

$P(b) = 1/11$	$P(d\|b) = 0.9$ $P(d\|\neg b) = 0.01$	$P(o\|d) = 0.9$ $P(o\|\neg d) = 0.2$

Now, we are supposed to compute the probability that the battery is low given the observer's report. I took this to mean (see the PS#8 FAQ) that we must compute both of

$$P(b|o) \text{ and } P(b|\neg o)$$

Starting with the first one, we obtain

$$\begin{aligned}
P(b|o) &= P(o|b)P(b)/P(o) \\
&= \alpha P(o|b)P(b) \\
&= \alpha \sum_D P(o|D)P(D|b)P(b) \\
&= \alpha P(b)[P(o|d)P(d|b) + P(o|\neg d)P(\neg d|b)] \\
&= \alpha(1/11)[(0.9)(0.9) + (0.2)(0.1)] \\
&= \alpha(83/1100)
\end{aligned}$$

The complementary probability (for normalization) is

$$\begin{aligned}
P(\neg b|o) &= \alpha P(\neg b)[P(o|d)P(d|\neg b) + P(o|\neg d)P(\neg d|\neg b)] \\
&= \alpha(10/11)[(0.9)(0.01) + (0.2)(0.99)] \\
&= \alpha(207/1100)
\end{aligned}$$

So, $P(b|o) = 83/(83 + 207) = \boxed{0.2862}$.

Similarly, one can compute $P(b|\neg o)$:

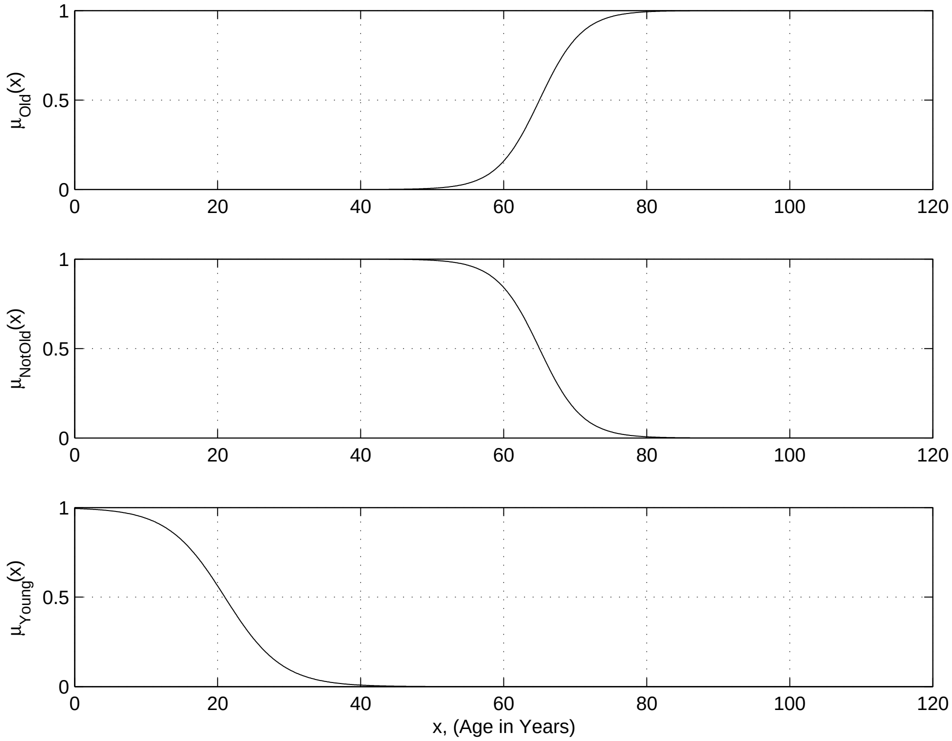
$$\begin{aligned}
P(b|\neg o) &= \alpha P(b)[P(\neg o|d)P(d|b) + P(\neg o|\neg d)P(\neg d|b)] \\
&= \alpha(1/11)[(0.1)(0.9) + (0.8)(0.1)] \\
&= \alpha(17/1100)
\end{aligned}$$

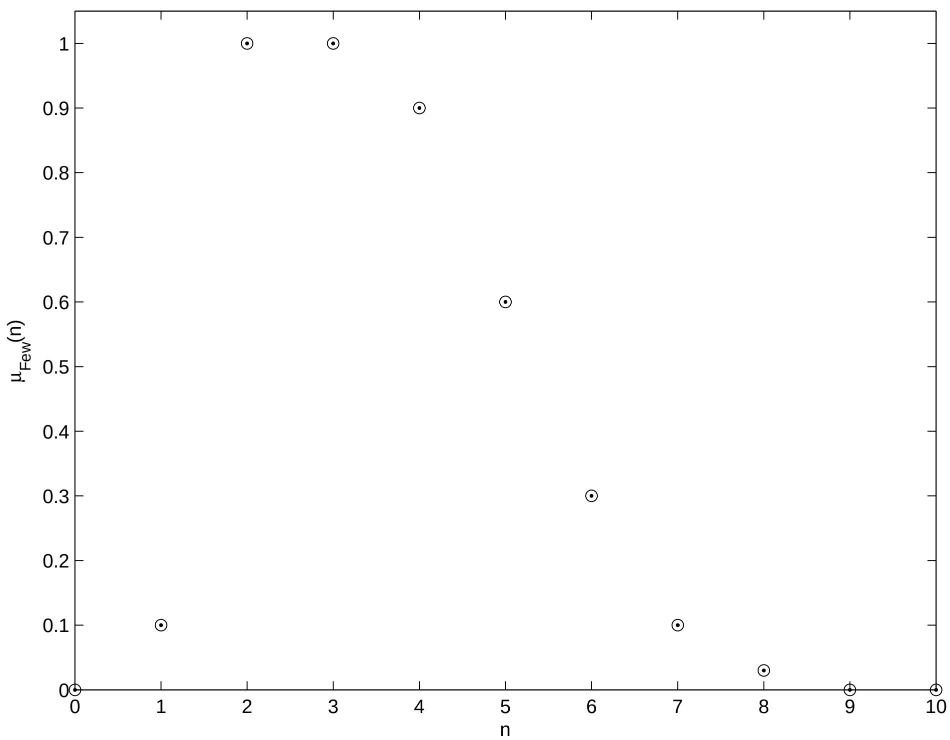
$$\begin{aligned}
P(\neg b|\neg o) &= \alpha P(\neg b)[P(\neg o|d)P(d|\neg b) + P(\neg o|\neg d)P(\neg d|\neg b)] \\
&= \alpha(10/11)[(0.1)(0.01) + (0.8)(0.99)] \\
&= \alpha(793/1100)
\end{aligned}$$

So, $P(b|\neg o) = 17/(17 + 793) = \boxed{0.0210}$.

Problem 8.5

The membership functions for the fuzzy sets in (a)–(d) appear below, respectively.





(e) Note "old" and "not old" are complements, but "not old" and "young" need not be the same.

Problem 8.6

The following experiment and results were obtained by former student, Stuart Morgan:

> **A Monte Carlo simulation with 1,000,000 trials gave the following outcome, which is consistent with the calculated results:**
>
> | A | B | C | D | Occurrences |
> |---|---|---|---|---|
> | 1 | 1 | 1 | 1 | 500145 |
> | 0 | 0 | 0 | 0 | 125003 |
> | 0 | 0 | 1 | 0 | 62640 |
> | 0 | 0 | 1 | 1 | 62481 |
> | 0 | 1 | 0 | 0 | 62570 |
> | 0 | 1 | 0 | 1 | 62472 |
> | 0 | 1 | 1 | 1 | 124689 |
>
> **Thus the empirical probability $P(A|D) = \frac{500145}{500145+62481+62472+124689} = 0.667$**

I wrote a program following the example Matlab code from class and obtained

$$\boxed{P(a|d) \approx 0.6660}$$

after just 10,000 trials. **The code has been suppressed.**

Problem 8.7

My former student, Stuart Morgan, wrote

> A Monte Carlo [MC; using direct or rejection sampling] verification would be possible, but not very practical, since even in one billion trials, this event would only be expected to happen about [69] times. However, the transformation
>
> $$P(\neg j, \neg m, b, e) = P(\neg j, \neg m|b, e)P(b, e)$$
>
> suggests a simply way to obtain a partial MC verification: assume in each trial that $B = E = True$ and then run MC simulations to find
>
> $$P(\neg j, \neg m|b, e) \tag{3}$$
>
> Then, multiplying by $P(E)P(B)$ would give a semi-empirical result.

This is just an intuitive explanation of the same answer that likelihood weighting (see Russel & Norvig, pp. 514–5) would have computed. I edited the example code given out in class to compute the probability in Equation (3) and found that

$$P(\neg j, \neg m|b, e) \approx 0.0342$$

after only 10000 trials. Hence,

$$P(\neg j, \neg m, b, e) \approx (0.001)(0.002)(0.0342) = \boxed{6.8440 \times 10^{-8}}$$

The code has been suppressed.