Lecture 23

Note Title 3/15/2005

Brownian Motion

1. $\{B(t) : t \geq 0\}$ has stationary indpt increments
2. $B(t)$ has normal distribution $\forall t \geq 0$
3. $E[B(t)] = 0$ $\forall t \geq 0$
4. $B(0) = 0$

$\Rightarrow$ $B(t+\delta) - B(t)$ normal w/ mean 0 and variance δ

Fractional Brownian Motion B_H

Depends on a parameter H (Hurst parameter)

$$\frac{1}{2} \leq H < 1$$

$H = \frac{1}{2} \Rightarrow$ Brownian motion

1. $B_H(t)$ continuos
2. $B_H(0) = 0$
3. $B_H(t+\delta) - B_H(t)$ normal w/ mean 0 and variance δ^{2H}

Remark: Stationary increments (do not depend on t)
Independent?

Proposition 1 $\quad E[B_H(t)] = 0 \quad \forall t$

Pf $\quad E[B_H(t)] = E[B_H(t) - B_H(0)] \underset{\text{axiom 3}}{=} 0$

$\square$

Proposition 2 $\quad Var[B_H(t)] = t^{2H}$

Pf $\quad Var[B_H(t)] = Var[B_H(t) - B_H(0)] = t^{2H}$

$\square$

Proposition 3 $\quad R(t_1, t_2) = E[B_H(t_1) B_H(t_2)] = \frac{1}{2}\left(t_1^{2H} + t_2^{2H} - |t_2 - t_1|^{2H}\right).$

Pf $\quad E[(B_H(t) - B_H(s))^2] = E[B_H^2(t) + B_H^2(s) - 2B_H(t)B_H(s)]$

$$E[B_H(t) B_H(s)] = \frac{1}{2}\left[E[B_H^2(t)] + E[B_H^2(s)] - E[(B_H(t) - B_H(s))^2]\right] =$$

$$= \frac{1}{2}\left[Var[B_H(t)] + Var[B_H(s)] - Var[B_H(t) - B_H(s)]\right] =$$

$$= \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

$\square$

Correlation between the increments $[-t, 0]$ and $[0, t]$

$$E[(B_H(0) - B_H(-t))(B_H(t) - B_H(0))] =$$

$$= E[-B_H(-t) B_H(t)] = -\frac{1}{2}\left((-t)^{2H} + t^{2H} - |-t-t|^{2H}\right) =$$

$$= \frac{1}{2}(2t)^{2H} - t^{2H} \neq 0 \quad \text{for } H \neq \tfrac{1}{2}.$$

$$\rho = \frac{E[XY] - E[X]E[Y]}{\sqrt{Var[X]\,Var[Y]}} \neq 0$$

Eventually: $A(t) =$ cumulative traffic from 0 and t

$$A(t) = mt + a B_H(t)$$

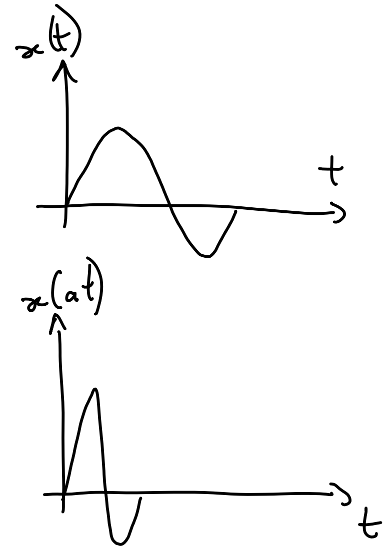
$$E[A(t)] = mt \qquad \frac{E[A(t)]}{t} = m$$

$$Var[A(t)] = a^2 Var[B_H(t)] = a^2 t^{2H}$$

Self-similarity

For continuous stochastic processes:

$$x(t) \overset{d}{=} \frac{1}{a^H} x(at)$$



1. $E[x(t)] = \dfrac{E[x(at)]}{a^H}$

2. $Var[x(t)] = \dfrac{Var[x(at)]}{a^{2H}}$

3. $R(t,s) = \dfrac{R(at, as)}{a^{2H}}$