by $x_j = x_j^+ - x_j^-$, and replace column A_j ... yields the LP

$$\min \hat{c}'\hat{x}$$
$$\hat{A}\hat{x} = b \qquad (3.\ldots$$
$$\hat{x} \geq 0$$

where

$$\hat{A} = \left[A_j, j \in N \middle| (A_j, -A_j),\ j \in \bar{N} \middle| \begin{matrix} 0, i \in M \\ -I, i \in \bar{M} \end{matrix} \right]$$

and

$$\hat{x} = \text{col}\,(x_j, j \in N | (x_j^+, x_j^-),\ j \in \bar{N} | x_i^s, i \in \bar{M})$$
$$\hat{c} = \text{col}\,(c_j, j \in N | (c_j, -c_j),\ j \in \bar{N} | 0)$$

We then know from the optimality criterion $\bar{c} \geq 0$ and the simplex algorithm that if there is an optimal solution $\hat{x}_0$ to (3.2), then there exists a basis $\hat{\mathscr{B}}$ for the LP in Eq. 3.2 such that

$$\hat{c}' - (\hat{c}_B'\hat{B}^{-1})\hat{A} \geq 0$$

Thus, $\pi' = \hat{c}_B'\hat{B}^{-1}$ is a feasible solution to the linear constraints

$$\pi'\hat{A} \leq \hat{c}' \qquad (3.3)$$

where $\pi \in R^m$, and m is the number of rows in the original A. These inequalities have three parts, depending on which set of columns of $\hat{A}$ is involved. The first set yields simply

$$\pi' A_j \leq c_j, \qquad j \in N \qquad (3.4)$$

The next set corresponds to the unconstrained x_j, $j \in \bar{N}$, and comes in pairs:

$$\begin{aligned} \pi' A_j &\leq c_j, \\ -\pi' A_j &\leq -c_j \end{aligned} \qquad j \in \bar{N}$$

which is equivalent to

$$\pi' A_j = c_j, \qquad j \in \bar{N} \qquad (3.5)$$

The final set corresponds to the inequalities $i \in \bar{M}$:

$$-\pi_i \leq 0 \qquad i \in \bar{M}$$

or

$$\pi_i \geq 0 \qquad i \in \bar{M} \qquad (3.6)$$

Equations 3.4, 3.5, and 3.6 define the constraints of a new LP, called the *dual* of the starting LP; the starting LP is called the *primal*. The value $\pi' = \hat{c}_B'\hat{B}^{-1}$ is feasible in the dual. If we define the cost function of the dual as max $\pi'b$, then π' is not only feasible, but optimal! We summarize this in the next definition and theorem.

...tion 3.1

...ven an LP in general form, called the *primal*, the *dual* is defined as ...:

Primal		Dual
$\min c'x$		$\max \pi'b$
$a_i'x = b_i$	$i \in M$	$\pi_i \gtrless 0$
$a_i'x \geq b_i$	$i \in \bar{M}$	$\pi_i \geq 0$
$x_j \geq 0$	$j \in N$	$\pi' A_j \leq c_j$
$x_j \gtrless 0$	$j \in \bar{N}$	$\pi' A_j = c_j$ □

...eorem 3.1 *If an LP has an optimal solution, so does its dual, and at optimality ...eir costs are equal.*

Proof Let x and π be feasible solutions to the primal and dual, respectively. Then

$$c'x \geq \pi' A x \geq \pi' b \qquad (3.7)$$

That is, the cost in the primal always dominates the cost in the dual. Since we assume the primal has a feasible solution, the dual cannot have a solution unbounded in cost. The dual has the feasible solution π' discussed above, so by the simplex algorithm, it has an optimum. We note that the cost of this π' is

$$\pi' b = \hat{c}_B'\hat{B}^{-1}b = \hat{c}_B'\hat{x}_0$$

which is the optimal cost in the primal. Therefore, by (3.7), this π' is optimal in the dual. □

An important feature of duality is the symmetry expressed in the following theorem.

Theorem 3.2 *The dual of the dual is the primal.*

Proof Write the dual as

$$\begin{aligned} &\min \pi'(-b) \\ (-A_j')\pi &\geq -c_j \qquad j \in N \\ (-A_j)\pi &= -c_j \qquad j \in \bar{N} \\ \pi_i &\geq 0 \qquad i \in \bar{M} \\ \pi_i &\gtrless 0 \qquad i \in M \end{aligned}$$

4. Where do we use the fact that we cl ... al prim algorithm?

5. Consider the primal-dual solution of sl ... optimal soluti RP that correspond to the optimal solı ... DRP given by Eq. 5.17. I for the suboptimal stages and the final optimal stage of the algorithm.

6. Suppose we allow negative arc weights in the node-arc formulation of sho path. Prove that the following conditions are equivalent.
   (a) There is a shortest s-t walk.
   (b) The dual LP is feasible.
   (c) There is no cycle with negative total cost.

7. Prove that in the primal-dual algorithm the cost of the feasible solution in increases by a positive amount during each iteration. Explain why this does n imply that the method terminates in a finite number of steps, as it would simplex.

8. Show that RP in max-flow and shortest path are highly degenerate.

NOTES AND REFERENCES

The primal-dual algorithm for general LP's was first described in

[DFF] DANTZIG, G. B., L. R. FORD, and D. R. FULKERSON, "A Primal-Dual Algorithm for Linear Programs," pp. 171–181 in *Linear Inequalities and Related Systems*, ed. H. W. Kuhn and A. W. Tucker. Princeton, N.J.: Princeton University Press, 1956.

It is presented there as a generalization of

[Ku] KUHN, H. W., "The Hungarian Method for the Assignment Problem," *Naval Research Logistics Quarterly*, 2, nos. 1 and 2 (1955), 83–97

and similar algorithms for more general flow problems, which will be described later in this book.

As mentioned in Sec. 5.4, primal-dual applied to shortest path leads to Dijkstra's algorithm:

[Di] DIJKSTRA, E. W., "A Note on Two Problems in Connexion with Graphs," *Numerische Mathematik*, 1 (1959), 269–71.

Primal-dual applied to max-flow leads to the Ford and Fulkerson algorithm, which is described in

[FF] FORD, L. R., JR., and D. R. FULKERSON, *Flows in Networks*. Princeton, N.J.: Princeton University Press, 1962.

This book, as Dantzig's, remains a standard almost 20 years after its appearance.

6

Primal-Dual Algorithms for Max-Flow and Shortest Path: Ford-Fulkerson and Dijkstra

6.1 The Max-Flow, Min-Cut Theorem

This chapter is devoted to the detailed development of the primal-dual algorithms derived in the last chapter for max-flow and shortest path. This is a point of transition in this book from the study of algorithms for general LP's to more specialized algorithms for certain network problems, all derived, as we have mentioned before, from primal-dual. We are thus headed towards algorithms that are in a sense less numerical and more combinatorial than general simplex algorithms. As we proceed, we shall also use linear programming theory to establish useful facts about a variety of graph-theoretic problems, beginning with the famous max-flow, min-cut theorem of the max-flow problem.

Returning to the node-arc formulation of max-flow in the previous chapter, consider the flow network $N = (s, t, V, E, b)$ with $n = |V|$ nodes and $m = |E|$ arcs; let the flow in arc (x, y) be denoted by $f(x, y)$. Then the max-flow problem is the following LP, which we think of as a dual:

$$\begin{aligned} &\max v \\ &Af + dv = 0 \\ &f \leq b \\ &f \geq 0 \end{aligned} \tag{6.1}$$

where $d \in R^n$ is defined, as before, by

$$d_i = \begin{cases} -1 & i = s \\ +1 & i = t \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

An important concept in dealing with flow problems in general, and one that played a central role in the invention of the Ford-Fulkerson algorithm, is that of a *cut*.

Definition 6.1

An *s-t cut* is a partition $(W, \bar{W})$ of the nodes of V into sets W and $\bar{W}$ such that $s \in W$ and $t \in \bar{W}$. The *capacity* of an *s-t* cut is

$$C(W, \bar{W}) = \sum_{\substack{(i,j) \in E \\ \text{such that} \\ i \in W, j \in \bar{W}}} b(i, j) \quad \square \tag{6.3}$$

Figure 6–1 illustrates the idea behind the definition of a cut; the capacity of a cut is the sum of the capacities of the "forward" arcs: those which go from nodes in W to nodes in $\bar{W}$. We would expect that the value of an *s-t* flow cannot exceed the capacity of an *s-t* cut, since all the *s-t* flow must pass through the forward arcs of a cut. This result is intimately related to the fact that cuts correspond to feasible solutions to the dual of the max-flow problem, which leads us next to write down the dual of the LP formulation of the max-flow problem in Eq. 6.1.

Assign variables $\pi(x)$ to the first n constraints, which correspond to flow conservation; and assign variables $\gamma(x, y)$ to the next m capacity constraints.

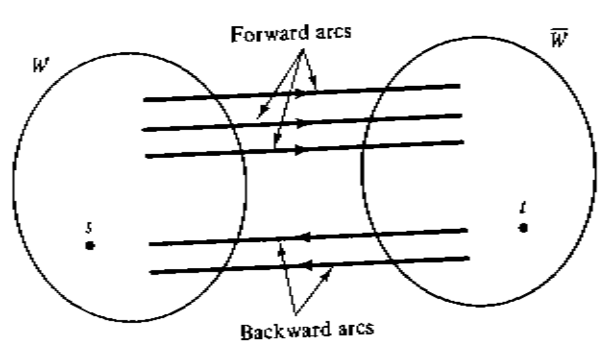

Figure 6–1 A cut in a flow network.

Since the first n constraints are equalities, $\pi(x) \gtrless 0$; and since the next m constraints are inequalities, $\gamma(x, y) \geq 0$. The LP in Eq. 6.1 is exactly in the form of the dual in Def. 3.1, so by symmetry, the primal in that definition is the dual for which we are looking:

$$\begin{aligned} \min \sum_{(x,y) \in E} &\gamma(x, y) b(x, y) \\ \pi(x) - \pi(y) + \gamma(x, y) &\geq 0 \qquad \text{for all } (x, y) \in E \\ -\pi(s) + \pi(t) &\geq 1 \\ \pi(x) &\gtrless 0 \\ \gamma(x, y) &\geq 0 \end{aligned} \tag{6.4}$$

The last inequality corresponds to the variable v.

We can now prove Theorem 6.1.

Theorem 6.1 *Every s-t cut determines a feasible solution with cost $C(W, \bar{W})$ to the dual of max-flow as follows:*

$$\begin{aligned} \gamma(x, y) &= \begin{cases} 1 & (x, y) \text{ such that } x \in W, y \in \bar{W} \\ 0 & \text{otherwise} \end{cases} \\ \pi(x) &= \begin{cases} 0 & x \in W \\ 1 & x \in \bar{W} \end{cases} \end{aligned} \tag{6.5}$$

Proof We need to check the inequality constraints in (6.4). There are four es to consider, since x and y may each be in W or $\bar{W}$. In each case the inequal- is easily verified, and the inequality is strict when and only when $x \in \bar{W}$ $y \in W$. Also, $\pi(s) = 0$ because $s \in W$, and $\pi(t) = 1$ because $t \in \bar{W}$; so

$$-\pi(s) + \pi(t) = 1$$

h verifies the last inequality. Finally, the cost of this dual feasible solution is

$$\sum_{(x,y) \in E} \gamma(x, y) b(x, y) = \sum_{\substack{(x,y) \in E \\ \text{such that} \\ x \in W, y \in \bar{W}}} b(x, y) = C(W, \bar{W}) \quad \square$$

rom Theorem 6.1, we get our main result.

6.2 (Max-flow, min-cut) *The value v of any s-t flow is no greater than* *city $C(W, \bar{W})$ of any s-t cut. Furthermore, the value of the maximum flow* *e capacity of the minimum cut, and a flow f and cut $(W, \bar{W})$ are jointly* *f and only if*

$$\begin{aligned} (x, y) &= 0 \qquad (x, y) \in E \text{ such that } x \in \bar{W} \text{ and } y \in W \\ (x, y) &= b(x, y) \qquad (x, y) \in E \text{ such that } x \in W \text{ and } y \in \bar{W} \end{aligned} \tag{6.6}$$

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$$f \leq b$$
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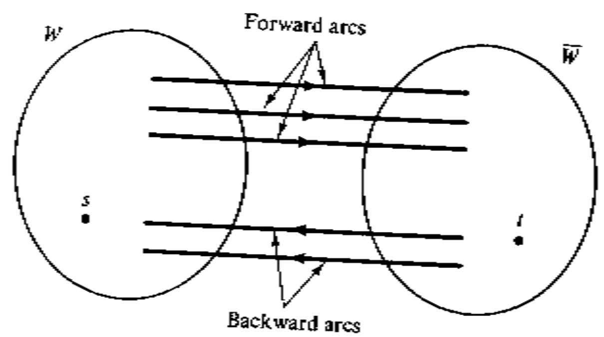

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