A New Technique for Solving Robot Calibration Equations with Partially Known Constraints

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Abstract

This paper presents an improved method for deducing kinematic parameters from constrained calibration data. The method assumes that the calibration data is constrained such that, for each recorded set of joint angles, the endpoint of the robot satisfies some parameterized mathematical constraint. Suitable mathematical constraint equations include points, lines, circles, spheres, planes, hyperplanes, and others. The constraint equations do not need to be fully known; some parameters of the constraints can be unknown, and the method can still be employed. The method prescribes a nonlinear transformation of the solution equations into a new space in which a vector function of parameters multiplies a vector function of joint variables. In this transformed space, the the vector function of parameters can be identified using a linear, least-squares approach. Subsequently, the transformed parameter-function vector is solved analytically for the actual parameter values. This approach differs from existing, iterative techniques in that there are no numerical problems of local-minima traps, slow convergence rates, null-space wandering or instability. Examples are presented to illustrate the method, and results are presented for calibration of an AdeptOne robot.

1 Introduction

Identifying kinematic parameters plays a vital role in the overall performance of Industrial Robots. Due to nonlinear nature of kinematic relations the governing equations of calibration are highly non-linear and their solution is usually complicated. In this paper a new method is introduced that will essentially break down the original problem in two considerably easier to solve parts. Using the fixed form of non-linearity inherent in the kinematic equation, a warping transformation is defined that will convert the forward kinematics to a linear transformation. Given the ease of use and broad availability of linear tools, one can use standard procedures to solve the warped form of the problem. Deducing the original physical parameters of the robot from the warped parameters will be the second part of the routine.

The paper is organized in four parts. The first part is a brief and general review of the present methods. In the second part the technique will be introduced and discussed in detail. Examples in the third section are to clarify the steps involved in the method. Finally results of the actual experiment that has been carried out on the AdeptOne robot are presented.

2 Overview

In this section the general form of Robot Calibration problem along with a brief description of the present methods of solution is presented. The kinematic equations will be the starting point. The constraint path is introduced next. Unknown parameters are considered for the constraint equation as well as the kinematic model.

The mapping between joint and task spaces, or *forward kinematics* can be written as:

$$\mathbf{x} = \mathbf{f}(\mathbf{q}, \mathbf{p}) \tag{1}$$

where $\mathbf{x}$, $\mathbf{q}$ and $\mathbf{p}$ are the pose of the end-effector, the joint variables and a vector of unknown calibration parameters respectively.

After establishing the structure of $\mathbf{f}$ or the kinematic model, three-dimensional coordinate measuring devices and techniques can be used at this stage to measure the end-effector pose while the joint variables are being recorded. Using this data one can solve for unknown $\mathbf{p}$ by minimizing a suitable penalty or error function [1, 2, 3].

Coordinate measurement, especially in three dimensions, is usually expensive and time consuming. To avoid direct measurement of the coordinates, the end-effector can

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be constrained to follow a particular path [4, 5, 6, 7]. The path is described by the functional relation:

$$\mathbf{g}(\mathbf{x}, \mathbf{p}_g) = \mathbf{0}$$

where $\mathbf{p}_g$ represents the unknown parameters of the path. Combining these two relations the constraint equation can be expressed in joint space:

$$\mathbf{g}(\mathbf{f}(\mathbf{q}, \mathbf{p}), \mathbf{p}_g) = \mathbf{0}$$

Many different algorithms have been suggested for various kinematic characteristics and constraining paths [8, 9, 7, 10, 11].

The common solution approach is to use a Taylor series expansion of the error function in the vicinity of the nominal calibration parameters, $\mathbf{p}_0$:

$$e^2 = e^2\big|_{\mathbf{p}_0} + \mathbf{k}^T(\mathbf{p} - \mathbf{p}_0) + \frac{1}{2}(\mathbf{p} - \mathbf{p}_0)^T \, \mathbf{L} \, (\mathbf{p} - \mathbf{p}_0) + \cdots$$

where:

$$\begin{aligned} \mathbf{k} &= \left.\frac{\partial e^2}{\partial \mathbf{p}}\right|_{\mathbf{p}_0} \\ \mathbf{L} &= \left.\frac{\partial^2 e^2}{\partial^2 \mathbf{p}}\right|_{\mathbf{p}_0} \end{aligned}$$

Typically, terms with orders higher than 2 are dropped. One problem that has been associated with this approach is determining the "observable" calibration parameters. Study of the **L** matrix has been suggested as means of achieving this goal [7, 1]. The null space of **L** has been proposed to be the set of the unidentifiable parameters. However the ignored higher-order terms in the series complicate determining the observable parameters.

In what follows, a new approach will be introduced. This approach avoids linearizing the error function by rearranging the governing equations of the problem. Its only limitation is that none of the transmission ratios for the joint variables can be a calibration parameter; in other words they are assumed to be perfectly known.

3 Development of Theory

A different approach for arranging the equations is presented that will result in a significantly simpler and robust process. The technique does not require initial estimates or iteration and there are, therefore no convergence rate and numerical stability concerns even in the presence of unobservable parameters.

If the joint variables can be measured accurately except for an unknown offset or "home" position, then the calibration parameters can be separated from the variables in the following form:

$$\mathbf{x} = \mathbf{M}(\mathbf{p})\mathbf{\Phi}(\mathbf{q}) \qquad (2)$$

elements of **M** are functions of only calibration parameters whereas components of vector $\mathbf{\Phi}$ are *linearly* independent functions of only joint variables. Basic column operations can be performed on **M** to find the minimum basis for the function space. $\mathbf{\Phi}$ will be referred to as the *Generalized Robot Variable* vector hereafter. Equation (2) simply states that the end-effector pose is a linear transformation of the generalized robot variables. In fact, if the transmission ratios of the revolute joints are known, this is always possible. If Denavit-Hartenberg notation [12] is used for the kinematic equations then, the "A" matrices are the building blocks of the kinematic equations. Introducing the null offsets or *home positions* (θ_{h_i}) for the joint variables a typical element of ${}^{i-1}\mathbf{A}_i$ matrix for a link is [13]:

$$a_{23} = -\cos(\theta_i - \theta_{h_i}) \sin\alpha_i$$

which after expanding the cosine function would be:

$$a_{23} = -\overbrace{\left[\cos\theta_{h_i} \, \sin\alpha_i\right]}^{\text{function of p only}} \cos\theta_i - \overbrace{\left[\sin\theta_{h_i} \, \sin\alpha_i\right]}^{\text{function of p only}} \sin\theta_i$$

every element of each **A** can be written as sums of terms in which the parameters are separated from the variables. Since the total kinematic equation is the product of such terms, arrangement (2) is possible.

As for the constraint equation imposed for obtaining the calibration data, some common cases have been studied and two of them will be presented here.

3.1 Case 1: Linear Form

The simplest and easiest endpoint constraint case is a hyperplane in task space which is the equivalent of a line in 2D or a plane in three dimensional space. This is also the most important case because it will lead to a very useful form which will play a key role in all other cases. Notice that the equation has been normalized to avoid redundant constants.

$$\mathbf{b}^T\mathbf{x} + 1 = 0$$

b is the vector that contains the hyperplane parameters-known or unknown. Substituting for **x** from (2):

$$\mathbf{c}^T\mathbf{\Phi} + 1 = 0 \qquad (3)$$

in which

$$\mathbf{c} \triangleq \mathbf{M}^T\mathbf{b}$$

the *Generalized Parameter* vector, $\mathbf{c}$, contains combinations of calibration parameters and parameters of the constraining path. With $\boldsymbol{\Phi}$ known for a number of configurations which satisfy the endpoint constraint equation one can solve for the elements of $\mathbf{c}$ using standard linear least-squares methods. If the number of distinct poses in the calibration data set is N, then $\mathbf{c}$ will be the solution to

$$\left(\sum_{\ell=1}^{N}\boldsymbol{\Phi}_\ell\boldsymbol{\Phi}_\ell^T\right)\mathbf{c} = -\sum_{\ell=1}^{N}\boldsymbol{\Phi}_\ell \tag{4}$$

with the error function defined as

$$e^2 \triangleq \sum_{\ell=1}^{N}(\mathbf{c}^T\boldsymbol{\Phi}_\ell + 1)^2$$

The value of $\boldsymbol{\Phi}$ when the robot is in its ℓth pose is denoted as $\boldsymbol{\Phi}_\ell$.

3.2 Case 2: Quadratic Form

In this section the case of quadratic constraint is considered. Examples of such constraints are circles, ellipses and parabolas in two dimensions and spheres and ellipsoids in 3D. It will be shown that these cases can be transformed into a case similar to previous case of a hyperplane constraint.

The general quadratic function for endpoint constraint can be written as

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + 1 = 0$$

substituting for $\mathbf{x}$ from (2):

$$\boldsymbol{\Phi}^T\mathbf{D}\boldsymbol{\Phi} + \mathbf{c}^T\boldsymbol{\Phi} + 1 = 0 \tag{5}$$

in which

$$\begin{aligned} \mathbf{c} &\triangleq \mathbf{M}^T\mathbf{b} \\ \mathbf{D} &\triangleq \mathbf{M}^T\mathbf{A}\mathbf{M} \end{aligned}$$

since the only concern is linear independence of the members of $\boldsymbol{\Phi}$, this form can be transformed into one similar to the linear case with some new definitions and substitutions. Assuming that $\mathbf{D}$ is in its equivalent symmetric form, define

$$\begin{aligned} \phi'_k &\triangleq \phi_i\phi_j \\ c'_k &\triangleq (2-\delta_{ij})D_{ij} \end{aligned} \tag{6}$$

where

$$\begin{aligned} k &= \frac{1}{2}(i-1)(2n-i)+j \\ & \quad n\text{: dimension of } \boldsymbol{\Phi} \\ \delta_{ij} &= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \end{aligned}$$

in this way (5) can be rewritten as:

$$\begin{pmatrix} \mathbf{c}' \\ \mathbf{c} \end{pmatrix}^T \begin{pmatrix} \boldsymbol{\Phi}' \\ \boldsymbol{\Phi} \end{pmatrix} + 1 = 0$$

or

$$\tilde{\mathbf{c}}^T\tilde{\boldsymbol{\Phi}} + 1 = 0 \tag{7}$$

this is the same form as (3). Given the linear independence of original $\boldsymbol{\Phi}$ the new function basis $\tilde{\boldsymbol{\Phi}}$ will have the same property. Equation (4) can be used to solve for $\tilde{\mathbf{c}}$.

Now the question is how can one extract the physical robot parameters from the $\mathbf{c}$ vector. In the cases that have been studied so far, we have found that the structure of the generalized parameter vector is such that one can solve for the desirable calibration parameters from $\mathbf{c}$ analytically without any major difficulty. In more complicated cases, symbolic math packages might be used to solve for the original parameters $\mathbf{p}$. Considering the fact that the above calculations have to be done only once for a particular kinematic structure, the complexity is justifiable. In the next section some examples are shown to further clarify the procedure.

The following steps summarize the algorithm:

1) Establish a kinematic model accounting for the unknown parameters. Equation (1).

2) Rearrange the nonlinear function as suggested by Equation (2). It is important that the generalized robot parameters be *linearly* independent. If not, perform the necessary elementary column operations to obtain a minimum function basis. $\mathbf{M}$ and $\boldsymbol{\Phi}$ are identified in this step.

3) Write your constraint equation. The constraint equation would preferably be in linear form. If not try to convert it to the linear form along the suggestions given in the quadratic form discussion.

4) Do the necessary substitutions so a linear or quadratic form can be achieved.

5) Perform the data acquisition with the robot confined to the constraint.

6) Solve for $\mathbf{c}$ using Equation (4).

7) Extract the physical robot parameters from $\mathbf{c}$. One should be able to do this by first attacking the less complex terms and solving from the base to the end-effector.

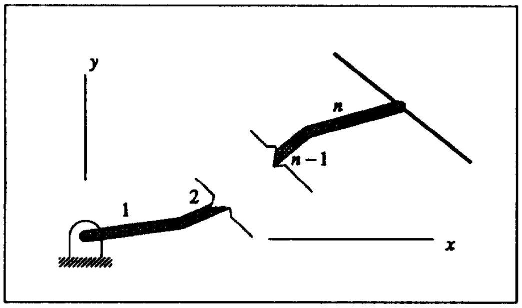

Figure 1: n link planar manipulator

4 Example Cases

To clarify the introduced procedure, two relatively small examples will be presented in this section. More practical examples (*e.g.* Spatial 5-joint articulated robot with straight line constraint) can be found in [14]. The first example will be an n-link planar manipulator, with an unknown straight line as a constraint. Later results of this example are applied to the AdeptOne robot. Second, the case of a 3-link planar manipulator with a point constraint is analyzed.

4.1 n-joint planar manipulator following a straight line

The robot is shown in Figure 1. Denavit-Hartenberg convention is used for kinematic parameters of each link. α's and d's are assumed to be zero. The goal is to identify the "observable" parameters and find their values.

1) Forward kinematics for the coordinates of the endpoint of the last link is written:

$$\begin{aligned} x &= \sum_{i=1}^{n} a_i \cos\Big(\sum_{j=1}^{i}(\theta_j - \theta_{h_j})\Big) \\ y &= \sum_{i=1}^{n} a_i \sin\Big(\sum_{j=1}^{i}(\theta_j - \theta_{h_j})\Big) \end{aligned}$$

2) After expanding the trigonometric functions and regrouping:

$$\mathbf{x} = \mathbf{M}\mathbf{\Phi}$$

where

$$\mathbf{M} = \begin{bmatrix} a_1 C_{h_1} & a_1 S_{h_1} & a_2 C_{h_{12}} & a_2 S_{h_{12}} & \cdots \\ -a_1 S_{h_1} & a_1 C_{h_1} & -a_2 S_{h_{12}} & a_2 C_{h_{12}} & \cdots \end{bmatrix}$$

$$\mathbf{\Phi} = \begin{pmatrix} C_1 \\ S_1 \\ C_{12} \\ S_{12} \\ \vdots \end{pmatrix}$$

in which

$$\begin{aligned} C_{1i} &= \cos\sum_{j=1}^{i}\theta_j & S_{1i} &= \sin\sum_{j=1}^{i}\theta_j \\ C_{h_{1i}} &= \cos\sum_{j=1}^{i}\theta_{h_j} & S_{h_{1i}} &= \sin\sum_{j=1}^{i}\theta_{h_j} \end{aligned}$$

3) An unknown straight line is the constraint in this case. The equation for such line can be written as

$$b_1 x + b_2 y + 1 = 0$$

b_1 and b_2 are assumed to be unknown.

4) the generalized calibration parameter vector $\mathbf{c}$ can be found from

$$\mathbf{c} = \mathbf{M}^T\mathbf{b}$$

or

$$\mathbf{c} = \begin{pmatrix} a_1 b_1 C_{h_1} - a_1 b_2 S_{h_1} \\ a_1 b_1 S_{h_1} + a_1 b_2 C_{h_1} \\ a_2 b_1 C_{h_{12}} - a_2 b_2 S_{h_{12}} \\ a_2 b_1 S_{h_{12}} + a_2 b_2 C_{h_{12}} \\ \vdots \end{pmatrix}$$

5, 6) It is assumed that the data has been collected for N configurations of the manipulator. The ℓth configuration being $\mathbf{\Phi}_\ell$. Equation (4) is formed and solved for $\mathbf{c}$.

7) In this step the physical parameters are extracted from $\mathbf{c}$. Defining ($\tan\beta \triangleq \frac{b_2}{b_1}$), $\mathbf{c}$ can be re-written as

$$\mathbf{c} = \begin{pmatrix} \frac{a_1 b_1}{\cos\beta}\cos(\beta + \theta_{h_1}) \\ \frac{a_1 b_1}{\cos\beta}\sin(\beta + \theta_{h_1}) \\ \frac{a_2 b_1}{\cos\beta}\cos(\beta + \theta_{h_1} + \theta_{h_2}) \\ \frac{a_2 b_1}{\cos\beta}\sin(\beta + \theta_{h_1} + \theta_{h_2}) \\ \vdots \end{pmatrix}$$

it can be seen that $(\beta + \theta_{h_1})$ always appears as one unit, so if β is not known θ_{h_1} can not be identified. It is also evident that the link lengths can be found if b_1 is known, otherwise only the ratios can be extracted.

The physical robot parameters are

$$\beta + \theta_{h_1} = \tan^{-1}\frac{c_2}{c_1}$$

$$\theta_{h_2} = \tan^{-1}\frac{c_4}{c_3} - (\beta + \theta_{h_1})$$

$$\vdots$$

$$\theta_{h_n} = \tan^{-1}\frac{c_{2n}}{c_{2n-1}} - (\beta + \sum_{j=1}^{n-1}\theta_{h_j})$$

$$\frac{a_2}{a_1} = \sqrt{\frac{c_3^2 + c_4^2}{c_1^2 + c_2^2}}$$

$$\frac{a_3}{a_1} = \sqrt{\frac{c_5^2 + c_6^2}{c_1^2 + c_2^2}}$$

$$\vdots$$

$$\frac{a_n}{a_1} = \sqrt{\frac{c_{2n-1}^2 + c_{2n}^2}{c_1^2 + c_2^2}}$$

4.2 3-link manipulator with fixed end-effector

This example shows how the procedure can be applied to a closed chain. The kinematics are the same as the above case for $n = 3$, so steps 1 and 2 are skipped.

3) The tip of the end-effector remains at a fixed point in the plane. Like the previous case the coordinates of this point is regarded as unknown. The constraint equation is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix}$$

4) The vector of generalized parameters, **c** is:

$$\mathbf{c} = \begin{pmatrix} \frac{a_1 S_{h_1}}{y_c} + \frac{a_1 C_{h_1}}{x_c} \\ -\frac{a_1 C_{h_1}}{y_c} + \frac{a_1 S_{h_1}}{x_c} \\ \frac{a_2 S_{h_{12}}}{y_c} + \frac{a_2 C_{h_{12}}}{x_c} \\ -\frac{a_2 C_{h_{12}}}{y_c} + \frac{a_2 S_{h_{12}}}{x_c} \\ \frac{a_3 S_{h_{123}}}{y_c} + \frac{a_3 C_{h_{123}}}{x_c} \\ -\frac{a_3 C_{h_{123}}}{y_c} + \frac{a_3 S_{h_{123}}}{x_c} \end{pmatrix}$$

Define $(\tan\beta \triangleq \frac{y_c}{x_c})$. Normalizing and simplifying the equation:

$$\mathbf{c} = \begin{pmatrix} 1 \\ \frac{-1}{\tan(\beta + \theta_{h_1})} \\ \frac{a_2}{a_1 \sin(\beta + \theta_{h_1})}\sin(\beta + \theta_{h_1} + \theta_{h_2}) \\ \frac{-a_2}{a_1 \sin(\beta + \theta_{h_1})}\cos(\beta + \theta_{h_1} + \theta_{h_2}) \\ \frac{a_3}{a_1 \sin(\beta + \theta_{h_1})}\sin(\beta + \theta_{h_1} + \theta_{h_2} + \theta_{h_3}) \\ \frac{-a_3}{a_1 \sin(\beta + \theta_{h_1})}\cos(\beta + \theta_{h_1} + \theta_{h_2} + \theta_{h_3}) \end{pmatrix}$$

5, 6) Data is assumed to be available for this case. Equation (4) is solved for **c**.

7) The physical parameters of the robot can be expressed in terms of **c**. Note that once again, if the coordinates of the constraint point (x_c, y_c) are not known θ_{h_1} and one length can not be identified.

$$\beta + \theta_{h_1} = \tan^{-1}\frac{1}{-c_2}$$

$$\theta_{h_2} = \tan^{-1}\frac{c_3}{c_4} - (\beta + \theta_{h_1})$$

$$\theta_{h_3} = \tan^{-1}\frac{c_5}{c_6} - (\beta + \theta_{h_1} + \theta_{h_2})$$

$$\frac{a_2}{a_1} = \sqrt{c_3^2 + c_4^2}$$

$$\frac{a_3}{a_1} = \sqrt{c_5^2 + c_6^2}$$

5 Experimental Setup and Results

The above principles are applied to the AdeptOne robot. The first two joints were used to imitate a simple 2-link planar case. The guidelines given in [4] were followed to set up the experiment. A laser beam was used to create an arbitrary straight line in robot's workspace. The feedback from a four quadrant optical detector at the last link's endpoint was used to accurately keep the end-effector on the constraint. Meanwhile the encoder readings were recorded. For the details of the equipment and experiment setup see [4].

Using the equation developed in section 4.1 for $n = 2$ the following results were obtained:

AdeptOne	nominal	computed(1)	computed(2)
a_2/a_1	1.13333	1.13323	1.13338
$\theta_{h_2}[rad]$	0	0.0446	0.0445
$err_{rms}[\mu m]$	–	32.20	35.68

In the above table err_{rms} is calculated as the deviation of the data points from a straight line after finding the observable unknowns:

$$err_{rms} = \sqrt{\frac{\min(\text{eigenvalue}(\mathbf{I}))}{N}}$$

where **I** is the inertia matrix about the center of the points:

$$\mathbf{I} = \begin{bmatrix} \sum_{\ell=1}^{N} \hat{y}_\ell^2 & -\sum_{\ell=1}^{N} \hat{x}_\ell \hat{y}_\ell \\ -\sum_{\ell=1}^{N} \hat{x}_\ell \hat{y}_\ell & \sum_{\ell=1}^{N} \hat{x}_\ell^2 \end{bmatrix}$$

$$\begin{aligned} \hat{x} &= x - \frac{1}{N}\sum_{\ell=1}^{N} x_\ell \\ \hat{y} &= y - \frac{1}{N}\sum_{\ell=1}^{N} y_\ell \end{aligned}$$

6 Closure

In this paper, a new technique for solving the resulting equations from Kinematic Calibration especially with incomplete knowledge of constraint equation was introduced and formulated. Based on the geometrical properties of the robot, generalized kinematic variables and parameters were defined. It was shown that the end-effector coordinates are linear mappings of the generalized variables. Different cases for the constraint equation were described and ways to reduce them to a form solvable by standard least-squares method were discussed. To further clarify the technique two simple yet complete examples were given. Finally the results of applying the proposed technique to the AdeptOne robot were reported.

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